WORKSHOP ON PROOF MINING

Technische Universität Darmstadt
September 4 – 6, 2024

Contents

https://sites.google.com/view/wpm24

Welcome Address

Proof theory started in the school of David Hilbert in Göttingen in the beginning of the 20th century with what is today known as Hilbert's program, a project in mathematical logic which aimed to show that the use of so-called ideal (i.e. non-constructive, set-theoretic or infinitary) principles in proofs of concrete so-called real statements could be (at least in principle) eliminated. In a modern view, this program is often subsumed under the goal of proving the consistency of powerful theories containing such ideal principles in certain finitistic theories. As is well-known, Gödel's second incompleteness theorem already rules out the provability of the consistency of many theories in themselves.

While Hilbert's program in this specific sense is therefore impossible to achieve globally, Georg Kreisel's work starting in the 1950's under the name of "unwinding proofs" marks the beginning of a seminal paradigm shift in proof theory, namely to use proof-theoretic devices developed in the course of Hilbert's program to extract computational information from specific given mathematical proofs, showing that Hilbert's program is actually largely achievable as far as most proofs from the literature are concerned. This field gained maturity in the 1990's through the work of Ulrich Kohlenbach, since then known as Proof Mining, and by now comprises over 200 works with many case studies situated within numerical mathematics and many branches of modern analysis.

Current research in this fast-growing area is concerned both with the development of new logical tools designed for facilitating a firm theoretical basis of this enterprise as well as with advancing the range of case studies into new territories relevant also for the mathematicians working in these fields in question.

With this meeting, we want to initiate a series of workshops dedicated to providing opportunities for people from the diverse areas connected to proof mining to meet, present work in progress and have an interdisciplinary exchange of ideas and knowledge. As such, the topics of interest in particular include logical aspects of proof interpretations as well as applied aspects from in particular nonlinear analysis and optimization.

The Organizing Committee
Ulrich Kohlenbach, Pedro Pinto, Nicholas Pischke

Outline of the workshop

The Workshop on Proof Mining 2024 is held on the campus of the Technische Universität Darmstadt from the 4th to the 6th of September 2024. The workshop takes place in person at the Room Vortragssaal (S1/20, R.01) in the University and State Library of Darmstadt. Funding for the workshop was provided by the Association for Symbolic Logic in the form of 3 travel grants that were awarded to graduate students and recent PhDs to attend the workshop; the DVMLG with sponsoring over 750 Euro that covered food and the accommodation of the room, the Department of Mathematics of the TU Darmstadt who provided further endowment and the DFG under the grant DFG KO 1737/6-2.

The program includes 23 talks from experts located within proof mining, its underlying theory and logical background as well as from experts in the fields of applications and can be found at the back of this booklet, with abstracts for talks following this outline.

The list of participants in particular includes the following 26 people ordered alphabetically:

Rob Arthan	Lemma 1 (UK)
Miroslav Bačák	University of Leipzig (DE)
Radu Boţ	University of Vienna (AT)
Horatiu Cheval	University of Bucharest (RO)
Bruno Dinis	University of Évora (PT)
Nicoleta Dumitru	University of Bucharest (RO)
Patrícia Engrácia	ISCTE Lisbon (PT)
Fernando Ferreira	University of Lisbon (PT)
Paulo Firmino	University of Lisbon (PT)
Anton Freund	University of Würzburg (DE)
Jesus Garcia-Falset	University of Valencia (ES)
Sorin-Mihai Grad	ENSTA Paris (FR)
Ulrich Kohlenbach	TU Darmstadt (DE)
Laurenţiu Leuştean	University of Bucharest (RO)
Morenikeji Neri	University of Bath (UK)
Adriana Nicolae	Babeș-Bolyai University (RO)
Paulo Oliva	Queen Mary University of London (UK)
Pedro Pinto	TU Darmstadt (DE)
Nicholas Pischke	TU Darmstadt (DE)
Thomas Powell	University of Bath (UK)
Shahram Saeidi	University of Kurdistan (IR)
Sam Sanders	Ruhr University Bochum (DE)
Lorenzo Sauras-Altuzarra	Vienna University of Technology (AT)
Andrei Sipoş	University of Bucharest (RO)
Alex Wan	University of Bath (UK)
Jin Wei	University of Pennsylvania (US)

Abstracts of Talks

Rates of convergence for stochastic processes

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I will present two ongoing projects in the new area of proof mining in stochastic optimization, each representing a distinct inroad into the area. First, I will give an account of recent work with Morenikeji Neri that focuses on convergence proofs based on martingales: This quantitative study of martingale convergence has resulted in several variants of the famous Robbins-Siegmund theorem that come equipped with numerical information, with applications including stochastic quasi-Fejér monotone sequences and gradient-descent type algorithms. Second, I will report on some work in progress with Nicholas Pischke on a new stochastic alternating Halpern-Mann scheme with noise terms: In addition to new convergence results (both qualitative and quantitative), this project has led to an interesting application in reinforcement learning, demonstrating that the techniques of proof mining are relevant for current research in the mathematics of artificial intelligence.

Fast continuous and discrete time methods for monotone equations

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In this talk we discuss continuous in time dynamics for the problem of approaching the set of zeros of a single-valued monotone and continuous operator. Such problems are motivated by minimax convex-concave and, in particular, by convex optimization problems with linear constraints. The central role is played by a second-order dynamical system that combines a vanishing damping term with the time derivative of the operator along the trajectory, which can be seen as an analogous of the Hessian-driven damping in cases where the operator originates from a potential. We demonstrate that the norm of the operator along the trajectory and the restricted gap function exhibit fast vanishing behaviour, and that the trajectory converges weakly to a solution of the monotone equation. The implicit and explicit discrete time models, enhanced with Nesterov's momentum and correcting terms, share the asymptotic features of the continuous dynamics ([1,2]). In the second part of the talk, we discuss the connection between the second-order dynamical system and a Tikhonov regularized first-order dynamical system, exhibiting fast convergence rates and strong convergence of the trajectory ([3]).

References

[1] R. I. Boţ, E.R. Csetnek, D.-K. Nguyen, Fast Optimistic Gradient Descent Ascent (OGDA) method in continuous and discrete time, *Foundations of Computational Mathematics*, DOI: 10.1007/s10208-023-09636-5, 2023.
[2] R. I. Boţ, D.-K. Nguyen, Fast Krasnosel'skii-Mann algorithm with a convergence rate of the fixed point iteration of $o(1/k)$, *SIAM Journal on Numerical Analysis* **61**(6), pp. 2813–2843, 2023.
[3] R. I. Boţ, D.-K. Nguyen, Tikhonov regularization of monotone operator flows not only ensures strong convergence of the trajectories but also speeds up the vanishing of the residuals, *arXiv:2406.00852*, 2024.

Bounding Interpretations with Abstract Spaces

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Early work on unifying functional interpretations has shown that the oldest functional interpretations (e.g. Gödel's Dialectica and Kreisel's modified realisability) only differ in the way they interpret the contraction axiom [1], which, in the setting of linear logic, even allows for these interpretations to be combined into so-called 'hybrid interpretations' [2]. Therefore, one can think of the choice of *interpreting contraction* as one of the degrees of freedom when designing a functional interpretation. More recently, a second degree of freedom has been studied in more detail, namely, the choice of how to *interpret quantifications over finite types* [3]. If we denote by $\forall \mathbf{x} \sqsubset_\tau \mathbf{a}\, A$ the choice of how contraction is interpreted, and by $x \prec^\tau a$ the interpretation of x being of type τ, then the different combinations can be summarised as follows (see [3] for complete table):

$x \prec^\tau a$	$\forall \mathbf{x} \sqsubset_\tau \mathbf{a}\, A$	**Interpretation**
$x =_\tau a$	$A[\mathbf{a}/\mathbf{x}]$	Dialectica
$x =_\tau a$	$\forall \mathbf{x}^\tau A$	Modified realizability
$x =_\tau a$	$\forall \mathbf{x} \leq^*_\tau \mathbf{a}\, A$	(*combination not sound*)
$x =_\tau a$	$\forall \mathbf{x} \in_\tau \mathbf{a}\, A$	Diller-Nahm
$x \leq^*_\tau a$	$A[\mathbf{a}/\mathbf{x}]$	(*combination not sound*)
$x \leq^*_\tau a$	$\tilde{\forall} \mathbf{x}^\tau A$	Bounded modified realizability
$x \leq^*_\tau a$	$\tilde{\forall} \mathbf{x} \leq^*_\tau \mathbf{a}\, A$	Bounded functional interpretation
$x \leq^*_\tau a$	$\tilde{\forall} \mathbf{x} \in_\tau \mathbf{a}\, A$	**Bounded Diller-Nahm**
$x \in_\tau a$	$A[\mathbf{a}/\mathbf{x}]$	Herbrand Dialectica ($\simeq$ Dialectica)
$x \in_\tau a$	$\forall \mathbf{x}^\tau A$	Herbrand realizability (for IL)
$x \in_\tau a$	$\tilde{\forall} \mathbf{x} \leq^*_\tau \mathbf{a}\, A$	**Herbrandized bfi**
$x \in_\tau a$	$\forall \mathbf{x} \in_\tau \mathbf{a}\, A$	Herbrand Diller-Nahm

Of particular note is the novel combination of majorizability and finite sets in the 'Bounded Diller-Nahm' interpretation:

$$
\begin{aligned}
|A \to B|^{\mathbf{f},\mathbf{g}}_{\mathbf{a},\mathbf{w}} &\equiv \tilde{\forall} \mathbf{b} \in \mathbf{gaw}\, |A|^{\mathbf{a}}_{\mathbf{b}} \to |B|^{\mathbf{fa}}_{\mathbf{w}} \\
|A \wedge B|^{\mathbf{a},\mathbf{v}}_{\mathbf{b},\mathbf{w}} &\equiv |A|^{\mathbf{a}}_{\mathbf{b}} \wedge |B|^{\mathbf{v}}_{\mathbf{w}} \\
|A \vee B|^{\mathbf{a},\mathbf{v}}_{\mathbf{b},\mathbf{w}} &\equiv \tilde{\forall} \mathbf{b}' \in \mathbf{b}\, |A|^{\mathbf{a}}_{\mathbf{b}'} \vee \tilde{\forall} \mathbf{w}' \in \mathbf{w}\, |B|^{\mathbf{v}}_{\mathbf{w}'} \\
|\exists x^\tau A|^{c,\mathbf{a}}_{\mathbf{b}} &\equiv \exists x \leq^*_\tau c \tilde{\forall} \mathbf{b}' \in \mathbf{b} |A|^{\mathbf{a}}_{\mathbf{b}'} \\
|\forall x^\tau A|^{\mathbf{f}}_{c,\mathbf{b}} &\equiv \forall x \leq^*_\tau c\, |A|^{\mathbf{f}c}_{\mathbf{b}}
\end{aligned}
$$

Motivated by the successful move in proof mining of working directly with 'abstract spaces' (e.g. [4, 5]) and the interpretation of the real numbers using the bounded functional interpretation [6], in (current) joint work, we are investigating an alternative approach to deal with abstract spaces X, by interpreting such quantifications directly as

$$
\begin{aligned}
|\exists x^X A|^{c,\mathbf{a}}_{\mathbf{b}} &\equiv \exists x \prec^X c\, \tilde{\forall} \mathbf{b}' \in \mathbf{b} |A|^{\mathbf{a}}_{\mathbf{b}'} \\
|\forall x^X A|^{\mathbf{f}}_{c,\mathbf{b}} &\equiv \forall x \prec^X c\, |A|^{\mathbf{f}c}_{\mathbf{b}}
\end{aligned}
$$

for suitable choices of the relation $x \prec^X c$. This talk aims to describe this novel approach and report on some preliminary results.

References

[1] P. Oliva, Functional interpretations of linear and intuitionistic logic, *Information and Computation*, **208**(5):565–577, 2010.
[2] P. Oliva, A. Hybrid functional interpretations of linear and intuitionistic logic, *Journal of Logic and Computation*, **22**(2):305–328, 2012.
[3] B. Dinis and P. Oliva, A parametrised functional interpretation of Heyting arithmetic, *Annals of Pure and Applied Logic*, **172**:4, 2021.
[4] U. Kohlenbach, Some logical metatheorems with applications in functional analysis. *Transactions of the American Mathematical Society*, **357**(1):89–128, 2005.
[5] P. Gerhardy and U. Kohlenbach. General logical metatheorems for functional analysis. *Transactions of the American Mathematical Society*, **360**:2615–2660, 2008.
[6] F. Ferreira, The abstract type of the real numbers, *Archive for Mathematical Logic*, **60** (7):1005–1017, 2021.

A subgradient-style method in Hadamard spaces

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Traditionally, Riemannian subgradient methods generally apply for a geodesically convex objective function and are described using tangent spaces and exponential maps. We diverge from this approach and discuss a subgradient-style algorithm in the setting of a general Hadamard space by considering objectives with horospherically convex lower level sets and by framing the iteration in the underlying space itself.

References

[1] A.S. Lewis, G. López-Acedo, A. Nicolae, Horoballs and the subgradient method, *arXiv:2403.15749 [math.OC]*.
[2] H. Zhang, S. Sra, First-order methods for geodesically convex optimization, *Journal of Machine Learning Research* **49**, pp. 1–22, 2016.

On the computational properties of fixed point methods with set-valued mappings

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This talk concerns ongoing research on the computational properties of fixed point methods applied to set-valued mappings. The first proper study of set-valued mappings dates back to the work of Nadler in [5], where the notion of set-valued contraction map was introduced and several fixed point results, in particular generalizing Banach's contraction principle, were established. Subsequently, several authors considered well-known fixed point methods adapted from the single-valued to the set-valued context (see for example [1,3,4,7-9]).

Our research aims at applying the techniques of proof mining to this setting, starting with the necessary logical background and some related metatheorems, in line with the groundbreaking work in the doctoral thesis of Pischke [6] (see also a first proof mining approach to the Hausdorff metric by Kohlenbach and Powell in [2]). As an example, we present the extraction of computational content from the convergence of the Mann iterative schema for a set-valued nonexpansive map.

We conclude with a discussion of future directions of research.

References

[1] J.S. Jung, Strong convergence theorems for multivalued nonexpansive nonself-mappings in Banach spaces, *Nonlinear Anal.* **66**, pp. 2345–2354, 2007.
[2] U. Kohlenbach, T. Powell, Rates of convergence for iterative solutions of equations involving set-valued accretive operators, *Computers & Mathematics with Applications* **80**(3) pp. 490–503
[3] T.C. Lim, A fixed point theorem for multivalued nonexpansive mappings in a uniformly convex Banach space, *Bull. Amer. Math. Soc.* **80** pp. 1123–1126, 1974.
[4] J.T. Markin, A fixed point theorem for set valued mappings, *Bull. Amer. Math. Soc.* **74**, pp. 639–640, 1968.
[5] S.B. Nadler Jr., Multi-valued contraction mappings, *Pacific J. Math.* **30**, pp. 475–487, 1969.
[6] N. Pischke, Proof-Theoretical Aspects of Nonlinear and Set-Valued Analysis, Ph.D. Thesis, TU Darmstadt, 2024.
[7] K.P.R. Sastry, G.V.R. Babu, Convergence of Ishikawa iterates for a multi-valued mapping with a fixed point, *Czechoslovak Math. J.* **55**, pp. 817–826, 2005.
[8] Y. Song, H. Wang, Convergence of iterative algorithms for multivalued mappings in Banach spaces, *Nonlinear Anal.* **70**, pp. 1547–1556, 2009.
[9] H-K. Xu, Multivalued nonexpansive mappings in Banach spaces, *Nonlinear Anal.* **43** pp.693 – 706, 2001.

Quantitative results on the subgradient extragradient method

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An essential problem in variational analysis and optimization is the Variational Inequality Problem (VIP), which consists in finding a point $u \in C$ such that

$$\langle f(u), x-u \rangle \geq 0 \quad \text{for all } x \in C,$$

where C is a nonempty, closed and convex subset of a Hilbert space H and $f : H \to H$.

In 1976, Korpelevich [6] introduced the extragradient method for solving the VIP in Euclidean spaces. Pischke [7] applied recently methods of proof mining developed by Kohlenbach, Leuştean and Nicolae [4] to compute, for the first time, uniform rates of metastability for the extragradient algorithm. Furthermore, rates of convergence are computed under additional metric regularity assumptions as defined by Kohlenbach, López-Acedo and Nicolae [5].

The extragradient algorithm involves performing two projections onto the set C for each iteration and this fact affects the efficiency of the algorithm when C is a general closed and convex set, as it requires solving two minimal distance problems to reach the next step. Censor, Gibali and Reich developed in [1,2] the subgradient extragradient method (SEM), an alternative approach that consists in replacing the second projection of the extragradient algorithm with a projection onto a specific subgradient half-space. SEM was defined in [1] for Euclidean spaces and generalized in [2] to Hilbert spaces. Furthermore, in [3], two modifications of SEM for solving the VIP in Hilbert spaces are explored.

In this talk, we discuss work in progress that consists in adapting Pischke's quantitative analysis of Korpelevich's extragradient algorithm to the subgradient extragradient method.

References

[1] Y. Censor, A. Gibali, S. Reich, Extensions of Korpelevich's extragradient method for the variational inequality problem in Euclidean space, *Optimization* **61**(9), pp. 1119–1132, 2012.
[2] Y. Censor, A. Gibali, S. Reich, The subgradient extragradient method for solving variational inequalities in Hilbert space, *J. Optim. Theory Appl.* **148**(2), pp. 318–335, 2011.
[3] Y. Censor, A. Gibali, S. Reich, Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space, *Optim. Methods Softw.* **26**(4-5), pp. 827–845, 2011.
[4] U. Kohlenbach, L. Leuştean, A. Nicolae, Quantitative results on Fejér monotone sequences, *Commun. Contemp. Math.* **20**(2), pp. 1750015, 2018.
[5] U. Kohlenbach, G. López-Acedo, A. Nicolae, Moduli of regularity and rates of convergence for Fejér monotone sequences, *Israel J. Math.* **232**, pp. 261–297, 2019.
[6] G. M. Korpelevich, The extragradient method for finding saddle points and other problems, *Èkon. Mat. Metody* **12**(4), pp. 747–756, 1976.
[7] N. Pischke, On Korpelevich's extragradient algorithm, *Z. Anal. Anwend.* **42**(1/2), pp. 117—129, 2023.

Proof theoretic aspects of the Laws of Large Numbers

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The Law of Large Numbers is a fundamental concept in probability theory. It states that as the sample size increases, the observed outcomes should get closer to the expected value. Kolmogorov is credited with putting this intuition into a rigorous mathematical setting. In this talk, we will explore how proof theory allows us to expand Kolmogorov's results by extracting quantitative data from various proofs. Additionally, we will show how a computability perspective can help us understand the quantitative nature of such results.

References

[1] M. Neri, Quantitative Strong Laws of Large Numbers, preprint, `arXiv:2406.19166`.

Sunny nonexpansive retractions in nonlinear spaces

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Undoubtedly, one of the most complicated instances of proof mining to date is the proof-theoretical unwinding of Reich's theorem, one of the most pivotal results in functional analysis, carried out by Kohlenbach and Sipoş in [1] for approximate curves associated with continuous pseudocontractive self-mappings in Banach spaces which are uniformly convex and uniformly smooth, and recently extended by Sipoş [4] to cover Reich's full convergence statement involving resolvents of accretive operators.

In this talk, we introduce the notion of a nonlinear smooth space [2], generalizing both CAT(0) spaces and smooth Banach spaces. We discuss how this notion allows for a unified treatment of several mathematical proofs in functional analysis and is suitable for proof mining metatheorems on the extraction of bounds. In particular, we show that Kohlenbach's and Sipos's treatment of Reich's result can be appropriately discussed in this nonlinear setting.

References

[1] U. Kohlenbach, A. Sipoş, The finitary content of sunny nonexpansive retractions. *Communications in Contemporary Mathematics*, **23**(1):1950093, 63 pp., 2021.
[2] P. Pinto, Nonexpansive maps in nonlinear smooth spaces. *Transactions of the American Mathematical Society*, **377**(9): 6379–6426, 2024.
[3] P. Pinto, The finitary content of sunny nonexpansive retractions in nonlinear spaces. Manuscript in preparation, 2024.
[4] A. Sipoş. On quantitative metastability for accretive operators. *Zeitschrift für Analysis und ihre Anwendungen*, **43**(3-4): 417–433, 2024.

Partial impredicativity and Fraïssé's conjecture

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Reverse mathematics provides a framework to compare the logical strength of axioms and theorems from various areas of mathematics. An astonishing number of theorems correponds to one of five axiom systems. At the same time, the gap between the strongest two axiom systems (ATR_0 and Π^1_1-CA_0) is populated by several relevant theorems. Inspired by the functional interpretation, H. Towsner has proposed a new axiom system within this gap [1]. Building on this, Y. Suzuki and K. Yokoyama have recently introduced a hierarchy that fills the entire gap [2]. Given a proof of a suitable theorem (namely of a Π^1_2-statement) in the strongest axiom system (Π^1_1-CA_0), one can – in a sense – apply proof mining to determine its place in the hierarchy. In my talk, I will explain this approach without assuming previous knowledge of reverse mathematics. I will also present a recent result on Fraïssé's conjecture, which is also known as Laver's theorem. The latter is at least as strong as ATR_0 and strictly weaker than Π^1_1-CA_0, as shown by R. Shore and A. Montalbán, respectively. I analyze the proof of Montalbán to show that Fraïssé's conjecture lies below the second level of the aforementioned hierarchy [3].

References

[1] H. Towsner, Partial impredicativity in reverse mathematics, *The Journal of Symbolic Logic* **72**(4), pp. 459–488, 2013.
[2] Y. Suzuki and K. Yokoyama, On the Π^1_2 consequences of Π^1_1-CA_0, preprint, `arXiv:2402.07136`.
[3] A. Freund, Fraïssé's conjecture, partial impredicativity and well-ordering principles, part I, preprint,`arXiv:2406.13485`.

Relaxed-inertial proximal point algorithms for problems involving strongly quasiconvex functions

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Introduced in the 1970's by Martinet for minimizing convex functions and extended shortly afterwards by Rockafellar towards monotone inclusion problems, the proximal point algorithm turned out to be a viable computational method for solving various classes of optimization problems even beyond the convex framework.

In this talk we propose a relaxed-inertial proximal point type algorithm for solving optimization problems consisting in minimizing strongly quasiconvex functions whose variables lie in closed convex sets. The method is then extended for equilibrium functions involving strongly quasiconvex functions. Computational results confirm the theoretical advances.

References

[1] S. M. Grad, F. Lara, R. T. Marcavillaca, Relaxed-inertial proximal point algorithms for nonconvex equilibrium problems with applications, *Journal of Optimization Theory*

and Applications, DOI: 10.1007/s10957-023-02375-1.
[2] S. M. Grad, F. Lara, R. T. Marcavillaca, Relaxed-inertial proximal point type algorithms for quasiconvex minimization, *Journal of Global Optimization* **85**, pp. 615-635, 2023.

On canonical functional interpretations of finite types

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When using the Diller-Nham interpretation to solve the problem of the contraction axiom, the uniform interpretation of the quantifiers (in intuitionistic logic) is

$$\{\exists x A(x)\}^{\boldsymbol{a}}_{\boldsymbol{B}} \;:\equiv\; \exists x \forall \boldsymbol{b} \in \boldsymbol{B}\, \{A(x)\}^{\boldsymbol{a}}_{\boldsymbol{b}}$$
$$\{\forall x A(x)\}^{\boldsymbol{a}}_{\boldsymbol{b}} \;:\equiv\; \forall x \{A(x)\}^{\boldsymbol{a}}_{\boldsymbol{b}}.$$

Here, the interpretation of A is $\exists \boldsymbol{a} \forall \boldsymbol{b} \{A\}^{\boldsymbol{a}}_{\boldsymbol{b}}$. The finite sets $\boldsymbol{B}$ of the existencial clause are needed because of the ∃-L rule. We let the uniform interpretation of the quantifiers be the background framework against which we endow our interpretations with computational meaning, via the consideration of computational informative types and predicates.

We consider two examples. Take the type N of natural numbers and put $\{x : N\}^k$ as being $x = k$. Computational significant quantifications are obtained from the background uniform quantifications by relativisation. In other words, we let $\exists x^N A(x)$ and $\forall x^N A(x)$ be, respectively, $\exists x(\{x : N\} \wedge A(x))$ and $\forall x(\{x : N\} \to A(x))$. A second example is to put $\{x : N\}^k$ as being $x \leq k$.

Given a computational interpretation for the ground type N, how should we define the computational interpretations of the finite types? In practice, there may be several possibilites. In this talk, we focus on a *canonical* manner of extending to finite types the interpretations of the ground types. The canonical manner takes $\{f : \sigma \to \tau\}$ to be $\forall x(\{x : \sigma\} \to \{fx : \tau\})$.

We first look at type 0 and type 1 quantifications. Regarding type 2, we see what it means for a functional Φ of type 2 to be realized (in the full set theoretic structure) according to the canonical manner. In the first example above, it means that Φ is a continuous functional with a modulus of pointwise continuity. In the second example, we apparently get a new notion. Both cases seem to define interesting structures of finite type functionals. It is also worth mentioning that the functional interpretation associated with the second case realises forms of uniform boundedness (collection) and contra-collection.

We may also briefly discuss abstract types, namely the type of the reals.

Do notice that this is work in progress, in need of proper checking and writing up.

References

[1] B. Dinis, P. Oliva, A parametrised functional interpretation of Heyting arithmetic, *Annals of Pure and Applied Logic* **172**, 24 pp., 2021.
[2] F. Ferreira, P. Oliva, Bounded functional interpretation, *Annals of Pure and Applied Logic* **123**(4), pp. 73–112, 2005.
[3] U. Kohlenbach, Applied Proof Theory: Proof Interpretations and their Use in Mathematics, *Springer Monographs in Mathematics*, Springer, xix + 532 pp., 2008.

Yosida approximations of the cumulative distribution function and applications in survival analysis

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Smoothing of the empirical cumulative distribution function (cdf) is often desirable when the underlying probability distribution is assumed to be continuous. Apart from kernel methods [1], which are used predominantly for this purpose, other types of smoothing have already been considered in the literature, for instance, approximations based on Bernstein polynomials [2], or piecewise linear approximations [3].

In the present work we explore an alternative method based on Yosida approximations, which is a classic regularization technique in maximal monotone operator theory. The Yosida approximation transforms a given cdf into a new cdf with better continuity properties, namely the new cdf is Lipschitz continuous, and its distance to the original cdf as well as its Lipschitz constant are both controlled by a parameter.

Our main results include the consistency and asymptotic unbiasedness of Yosida approximations as statistical estimators. In the spirit of proof mining we show how these properties depend on a modulus of uniform continuity of the underlying probability distribution.

We also present possible applications to survival analysis and pose two open problems in order to stimulate further research along these lines.

Part of the talk is a brief introduction into survival analysis in order to accommodate the needs of the audience.

References

[1] R. Servien, Estimation de la fonction de répartition: revue bibliographique, *Journal de la Societé Française de Statistique* **150**(2), pp. 84–104, 2009.
[2] A. Leblanc, On estimating distribution functions using Bernstein polynomials, *Ann. Inst. Statist. Math.* **64**(5), pp. 919–943, 2012.
[3] D. Blanke, D. Bosq, Polygonal smoothing of the empirical distribution function, *Stat. Inference Stoch. Process.* **21**(2), pp. 263–287, 2018.

Quantitative Analysis of Dvoretzky's Theorem in Stochastic Approximation Theory

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In the Autumn of 2022, inspired by work of Vajjha et al. on formalising a theorem of Dvoretzky in Coq [5], we became interested in mining the very neat proof they found in work of Derman and Sacks [2] from the 1950s for a quantitative version of Dvoretzky's qualitative result [3, 4]. The theorem deals with a sequence of real random variables $X_{n+1} = T_n(X_1, \ldots, X_n) + Y_n(X_1, \ldots, X_n)$ where T_n represents a process that we are trying to approximate, by sampling, and the random variable Y_n is an error term. Under suitable assumptions, including assumptions on the boundedness of $|T_n|$ and on the convergence of the sum $\sum_n \mathbb{E}(Y_n^2)$, the theorem states that X_n converges as $n \to \infty$ w.p.1. The bound on T_n is defined in terms of three sequences α_n, β_n and γ_n of non-negative reals such that

$\alpha_n \to 0$, $\sum_n \beta_n < \infty$ and $\sum_n \gamma_n = \infty$. In our quantitative formulation, the conclusion has the form

$$\forall \varepsilon, \delta(\mathbb{P}[\forall i \geq \Phi(\varepsilon, \delta)(|X_i| \leq \varepsilon)] \geq 1 - \delta)$$

where Φ is effectively computable in terms of the various rates of convergence and divergence of the sequences and series in the assumptions.

The proof involves Abel's criterion, Chebyshev's inequality, a form of the Kolmogorov inequality, the first Borel-Cantelli lemma and a lemma, specific to Dvoretzky's theorem, of Derman and Sacks concerning five sequences of real numbers showing the convergence of one of the sequence given various assumptions about the other four. Apart from the Borel-Cantelli, which we have covered in earlier work [1], we had to give quantitative versions of all of these as part of the present work. Perhaps surprisingly, these quantitative results are all direct: we have not needed to use metastability. Nonetheless, the results are quite complex: Φ is a very involved function of its inputs.

Dvoretzky's result has numerous applications in stochastic approximation, e.g., the original result in the subject due to Robbins and Monro. We are looking at extracting quantitative bounds for some of these applications using our quantitative analysis of Dvoretzky's theorem.

References

[1] R. Arthan, P. Oliva, On the Borel-Cantelli Lemmas, the Erdős-Rényi Theorem, and the Kochen-Stone Theorem, *Journal of Logic and Analysis* **13**(6), pp. 1–23, 2021.
[2] C. Derman, J. Sacks, On Dvoretzky's stochastic approximation theorem, *Ann. Math. Stat.* **30**, pp. 601–606, 1959.
[3] A. Dvoretzky, On stochastic approximation, *Berkeley Symp. on Math. Statist. and Prob.* **3**, pp. 39–55, 1956.
[4] A. Dvoretzky, Stochastic approximation revisited, *Adv. Appl. Math.* **7**, pp. 220–227, 1986.
[5] K. Vajjha, B. Trager, A. Shinnar, V. Pestun, Formalization of a Stochastic Approximation Theorem, In: *13th International Conference on Interactive Theorem Proving (ITP 2022), Leibniz International Proceedings in Informatics (LIPIcs)* **237**, pp. 31:1–31:18, 2022.

The alternating Halpern-Mann iteration for families of maps

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In recent joint work with Pinto [8], we generalize the alternating Halpern-Mann iteration, originally introduced by Dinis and Pinto [6], to countably infinite families of nonexpansive maps and prove its strong convergence towards a common fixed point of the families of maps within the context of Hadamard spaces. This work was motivated by the recent convergence results by Boţ and Meier [1] as well as the connected proof mining studies by Cheval [2, 3].

In this talk we present our approach, which follows and extends that of Dinis and Pinto [6]. Our results are obtained with methods arising from the research program of proof mining, which aims to analyse proofs in order to obtain new computational information. We moreover obtain a novel infinitary result: the strong convergence of the iteration. The proof relies on finitary arguments, avoiding the use of sequential weak compactness or other mathematically complicated arguments common in the setting of nonlinear geodesic spaces. In particular, we also obtain computable rates of asymptotic regularity in UCW-hyperbolic spaces and a computable metastability rate in CAT(0) spaces. These rates are uniform, depending only on bounds on the parameters and on quantitative information on the conditions. In CAT(0) spaces, we present linear and quadratic uniform rates of asymptotic regularity under certain conditions.

References

[1] R. I. Boţ, D. Meier, A strongly convergent Krasnosel'skiĭ-Mann-type algorithm for finding a common fixed point of a countably infinite family of nonexpansive operators in Hilbert spaces, *Journal of Computational and Applied Mathematics* **395**, No. 113589, 12pp, 2021.
[2] H. Cheval, Rates of asymptotic regularity of the Tikhonov-Mann iteration for families of mappings, arXiv:2304.11366v2, 2023
[3] H. Cheval, Quantitative metastability of the Tikhonov-Mann iteration for countable families of mappings, arXiv:2406.03429v1, 2024
[4] H. Cheval, U. Kohlenbach, L. Leuştean, On Modified Halpern and Tikhonov–Mann Iterations, *Journal of Optimization Theory and Applications* **197**(1), pp. 233–251, 2023
[5] H. Cheval, L. Leuştean, Quadratic rates of asymptotic regularity for the Tikhonov-Mann iteration, *Optimization Methods & Software*, **37**(6), pp. 2225–2240, 2022
[6] B. Dinis, P. Pinto, Strong convergence for the alternating Halpern-Mann iteration in CAT(0) spaces, *SIAM Journal on Optimization*, **33**(2), pp. 785–815, 2023
[7] F. Ferreira, L. Leuştean, P. Pinto, On the removal of weak compactness arguments in proof mining, *Advances in Mathematics*, **354**, 106728, 55, 2019
[8] P. Firmino, P. Pinto, The alternating Halpern-Mann iteration for families of maps, *(submitted), Oberwolfach Preprint OWP-2024-07*, 2024.
[9] U. Kohlenbach, Applied Proof Theory: Proof Interpretations and their Use in Mathematics, Springer Science & Business Media, 2008

Zero-one laws: A quantitative insight

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Zero-one laws are theorems that state that events of a certain type occur with probability 0 or 1. They are fundamental results which have applications in various areas of mathematics, including graph theory. In this talk, I will first reformulate the Borel zero-one law into a computational problem, before stating a specialized result that is linked closely with the Borel-Cantelli lemmas, recently studied by Arthan and Oliva in [1]. I will then take this as a starting point for a more general framework for studying zero-one laws computationally, before demonstrating some applications using this framework, including the zero-one law associated with bond percolation.

References

[1] R. Arthan, P. Oliva, On the Borel-Cantelli Lemmas, the Erdős-Rényi Theorem, and the Kochen-Stone Theorem, *Journal of Logic and Analysis* **13**(6), pp. 1–23, 2021.

Some results in reverse mathematics inspired by proof mining

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The study of (compact) metric spaces in second-order Reverse Mathematics (RM hereafter) is fundamentally based on *separability conditions*, while the latter are generally avoided in proof mining to enable the extraction of good computational data. Inspired by this observation, we study basic properties of 'unrepresented' compact metric spaces in Kohlenbach's higher-order RM, i.e. we do not assume separability conditions. Our results are four-fold as follows, each building on the next.

- Most definitions of compactness yield third-order theorems not provable from second-order comprehension axioms. Only one very specific choice of compactness definitions yields equivalences involving the Big Five of second-order RM.
- Many basic properties of compact metric spaces inhabit the range of *hyperarithmetical analysis*. Until recently, few natural examples of the latter were known.
- Some basic properties of compact metric spaces, like the intermediate value theorem, are equivalent to countable choice as studied in higher-order RM, namely $\mathsf{QF\text{-}AC}^{0,1}$.
- Some basic properties of compact metric spaces, like *a continuous function has a supremum* and *a countable set has measure zero*, imply strong axioms including Feferman's projection principle, full second-order arithmetic, and Kleene's quantifier $(\exists^3)$.

In conclusion, the removal of separability conditions from compact metric spaces results in rather strong theorems.

Products of hyperbolic spaces

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The class of uniformly smooth hyperbolic spaces was recently introduced by Pinto [1] as a common generalization of CAT(0) spaces and uniformly smooth Banach spaces, for which Reich's theorem on resolvent convergence may be proven. We define products of such spaces, showing that they are reasonably well-behaved. In this way, we provide the first example of a space for which Reich's theorem holds and which is neither a CAT(0) space, nor a convex subset of a normed space.

References

[1] P. Pinto, Nonexpansive maps in nonlinear smooth spaces. *Transactions of the American Mathematical Society*, `https://doi.org/10.1090/tran/9166`, 47 pp., 2024.

Fixed points of continuous mappings on unbounded domains

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Our purpose in this talk is to discuss under what conditions a nonexpansive or condensing mapping has a fixed point whether its domain is an unbounded subset of a Banach space and its range is not necessarily contained into its domain.

References

[1] Garcia-Falset J., *Fixed points of continuous pseudocontractive mappings on convex, closed unbounded domains*, Proc. Amer. Math. Soc. **152** (7) (2024), 2863–2876.
[2] Garcia-Falset J.; Muñiz-Pérez O., *Projected dynamical systems on Hilbert spaces,* J. Nonlinear and Convex Analysis **15** (2), (2014), 325–244..

Proof mining and asymptotic regularity

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Asymptotic regularity is a very important property in nonlinear analysis and optimization, introduced in the 1960s by Browder and Petryshyn [1] for the Picard iteration and extended to general iterations by Borwein, Reich, and Shafrir [2]. In numerous proofs of the strong or weak convergence of a nonlinear iteration, one obtains as an intermediate step the asymptotic regularity of the iteration, usually in a very general setting.

In this talk we present recent applications of proof mining providing effective uniform rates of asymptotic regularity for different nonlinear iterations associated to nonexpansive mappings (Tikhonov-Mann iteration [3], modified Halpern iteration [4], and the alternating Halpern-Mann iteration [5]), as well as for the viscosity approximation method (VAM) for m-accretive operators in Banach spaces [6] and a generalization of VAM to W-hyperbolic spaces [7]. Furthermore, we apply a lemma on sequences of real numbers due to Sabach and Shtern [8] to compute linear rates of asymptotic regularity for these iterations. We show in [9] that Sabach and Shtern's lemma can be used to obtain linear rates of asymptotic regularity for other Halpern-type iterations studied in optimization and nonlinear analysis.

References

[1] F. Browder, W. Petryshyn, The solution by iteration of nonlinear functional equations in Banach spaces, *Bull. Amer. Math. Soc. (N.S.)* **72**, pp. 571–575, 1966.
[2] J. Borwein, S. Reich, I. Shafrir, Krasnoselski-Mann iterations in normed spaces, *Canad. Math. Bull.* **35**, pp. 21–28, 1992.
[3] H. Cheval, L. Leuştean, Quadratic rates of asymptotic regularity for the Tikhonov-Mann iteration, *Optim. Methods Softw.* **37**, pp. 2225–2240, 2022.
[4] H. Cheval, U. Kohlenbach, L. Leuştean, On modified Halpern and Tikhonov-Mann iterations, *J. Optim. Theory Appl.* **197**, pp. 233-251, 2023.

[5] L. Leuştean, P. Pinto, Rates of asymptotic regularity for the alternating Halpern-Mann iteration, *Optim. Lett.* **18**, pp. 529–543, 2024.
[6] P. Firmino, L. Leuştean, Quantitative asymptotic regularity of the VAM iteration with error terms for m-accretive operators in Banach spaces, *Z. Anal. Anwend.*, DOI: 10.4171/ZAA/1772, 2024.
[7] P. Firmino, L. Leuştean, A generalization of the viscosity approximation method, work in progress, 2024.
[8] S. Sabach, S. Shtern, A first order method for solving convex bilevel optimization problems, *SIAM J. Optim.* **27**, pp. 640–660, 2017.
[9] H. Cheval, L. Leuştean, Linear rates of asymptotic regularity for Halpern-type iterations, *Math. Comp.*, DOI: 10.1090/mcom/3991, 2024.

Baaz's generalization method and its applications to number theory

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In [1] Matthias Baaz introduced a new technique of proof mining and, as an application, a property of the factors of Fermat numbers. In [2] I showed a geometric characterization of such factors, generalizing Baaz's pioneer result, and a related conjecture about point-lattices. After presenting this last problem at several conferences, Mabud Sarkar and René Schoof obtained partial answers, Vasile Brînzănescu observed a relation with the theory of unimodular matrices, which led to a joint theorem with Gergely Harcos, and Daniele Parisse revealed a connection with the theory of generalized Pillai equations. In this talk I will explain Baaz's method and display its aforementioned outcomes, which were included in [3].

References

[1] M. Baaz, Note on the generalization of calculations, *Theoretical Computer Science* **334**, 1999.
[2] L. Sauras Altuzarra, Some applications of Baaz's generalization method to the study of the factors of Fermat numbers, *Journal of Logic and Computation*, 2022.
[3] L. Sauras Altuzarra, From Logic to Discrete Geometry via Lattices, *doctoral thesis*, 2023.

Effective Rates for Nonlinear Ergodic Theorems and Differential Inclusions

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In this talk, we discuss the existence of computable rates of asymptotic regularity for Cesàro means for mappings with a modulus of regularity in a general setting. We also examine the computable rates of convergence of Cesàro means as well as the vector valued integrals with respect to means on semigroups, based on [1-5]. Additionally, we investigate the existence and uniqueness of solutions to a general second order differential inclusion of accretive type, which is new in Banach spaces. Furthermore, for the semigroup generated by these solutions, based on the results in [6, 7], we present a quantitative result in the form of a rate of convergence that depends, among other elements, on a modulus for the convergence condition.

References

[1] U. Kohlenbach, L. Leustean, A. Nicolae, Quantitative results on Fejér monotone sequences, *Communications in Contemporary Mathematics* **20** (2018), 42 pp.
[2] U. Kohlenbach, L. Lopez-Acedo, A. Nicolae, Moduli of regularity and rates of convergence for Fejér monotone sequences, *Israel Journal of Mathematics* **232** (2019), 261–297.
[3] A. Freund, U. Kohlenbach, R.E. Bruck, proof mining and a rate of asymptotic regularity for ergodic averages in Banach spaces, *Applied Set-Valued Analysis and Optimization* **4** (2022), 323–336.
[4] A. Freund, U. Kohlenbach, Bounds for a nonlinear ergodic theorem for Banach spaces, *Ergodic Theory and Dynamical Systems* **43** (2023), 1570–1593.
[5] R.E. Bruck, On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces, *Israel J. Math.* 38 (1981), 304–314.
[6] P. Pinto, N. Pischke, On computational properties of Cauchy problems generated by accretive operators, *Documenta Mathematica* **28** (2023), 1235–1274.
[7] N. Pischke, Rates of convergence for the asymptotic behavior of second-order Cauchy problems, *J. Math. Anal. Appl.* **533** (2024), 128078.

New horizons in (quantitative) Fejér monotonicity

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I discuss recent results [1], derived from the work [2], which provide quantitative and abstract strong convergence results for sequences from a compact metric space satisfying a certain form of generalized Fejér monotonicity where the metric can be replaced by a much more general type of function measuring distances (including, in particular, certain Bregman distances). These novel convergence results are established using a preceding finitary and quantitative theorem which constructs a rate of metastability for the Cauchy property of such sequences.

Utilizing these results, I then discuss some aspects of the recent work [3] where, in the context of general hyperbolic metric spaces, a new notion of a dual system (akin to the influential notion from the context of topological vector spaces) is introduced that allows for a uniform study of different notions of duality for these nonlinear spaces. Over this abstract notion of duality, I discuss how various notions from convex analysis can be lifted into this nonlinear setting, including Fréchet differentiability and Bregman distances. In particular, using a notion of a monotone operator relative to a given dual system and, using the new Fréchet derivatives, I study corresponding resolvents relative to a given gradient, generalizing the seminal notion of Eckstein from the linear setting. As a main application of the preceding general results, I then prove a convergence result of an analogue of the proximal point algorithm (together with quantitative results on its convergence). In the end, I discuss current work in progress on expanding the general and abstract (quantitative) convergence results by allowing for stochastic aspects.

References

[1] N. Pischke, Generalized Fejér monotone sequences and their finitary content, *Optimization*, to appear, 68pp., doi:10.1080/02331934.2024.2390114, 2024.
[2] U. Kohlenbach, L. Leuştean, A. Nicolae, Quantitative results on Fejér monotone sequences, *Communications in Contemporary Mathematics* **20**, 42pp., doi:`10.1142/S0219199717500158`, 2018.
[3] N. Pischke, Duality, Fréchet differentiability and Bregman distances in hyperbolic spaces, *Israel Journal of Mathematics*, to appear, 40pp., manuscript at `https://sites.google.com/view/nicholaspischke/notes-and-papers`, 2024.

Proof Mining in the Context of Strongly Nonexpansive Mappings

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In [1] we gave a quantitative form of the fundamental notion (due to Bruck and Reich) of strongly (quasi-)nonexpansive mappings (S(Q)NE) in terms of a so-called S(Q)NE-modulus. This approach, suggested by proof mining, was used to extract rates of asymptotic regularity for iterations of compositions of firmly nonexpansive ([2]), averaged ([8]) and supercoercively strongly nonexpansive ([9]) mappings. It also led - when further generalized to allow for 'quantitative quasiness' ([8]) - to rates of metastability for Halpern-type proximal point algorithms in Banach spaces which are both uniformly convex and uniformly smooth ([3]), CAT(0)-spaces ([8]), uniformly convex and smooth hyperbolic spaces ([5]) as well as the Bregman version of the Halpern-type proximal point algorithm in (suitable) Banach spaces ([6]). After briefly surveying some of these results, we give (joint work with J. Treusch) a rate of convergence for the Peaceman-Rachford splitting algorithm in the case where both operators are uniformly monotone using a combination of a recent result from [4] and a quantitative rate from [1].

References

[1] U. Kohlenbach, On the quantitative asymptotic behavior of strongly nonexpansive mappings in Banach and geodesic spaces, *Israel Journal of Mathematics* **216**, pp. 215–246, 2016.
[2] U. Kohlenbach, A polynomial rate of asymptotic regularity for compositions of projections in Hilbert space, *Foundations of Computational Mathematics* **19**, pp. 83–99, 2019.
[3] U. Kohlenbach, Quantitative analysis of a Halpern-type Proximal Point Algorithm for accretive operators in Banach spaces, *Journal of Nonlinear and Convex Analysis* **21**(9), pp. 2125–2138, 2020.
[4] L. Liu, W.M. Moursi, J. Vanderwerff, Strongly nonexpansive mappings revisited: uniform monotonicity and operator splitting, *SIAM Journal on Optimization* **33**(9), pp. 2570–2597, 2023.

[5] P. Pinto, Nonexpansive maps in nonlinear smooth spaces, *Transactions of the American Mathematical Society*, to appear, 47pp., 2024.
[6] N. Pischke, U. Kohlenbach, Effective rates for iterations involving Bregman strongly nonexpansive operators, submitted, 42pp., 2024.
[7] A. Sipoş, Abstract strongly convergent variants of the proximal point algorithm, *Computational Optimization and Applications* **83**, pp. 349–380, 2022.
[8] A. Sipoş, Quantitative inconsistent feasibility for averaged mappings, *Optimization Letters* **16**, pp. 1915–1925, 2022.
[9] A. Sipoş, The computational content of super strongly nonexpansive mappings and uniformly monotone operators, *Israel Journal of Mathematics*, to appear, 17pp., 2024.

4-6 September 2024
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Program

	WEDNESDAY	THURSDAY	FRIDAY
09:00 - 09:15	Opening Address		
09:15 - 10:05	**Powell**	**Freund**	**Sipoş**
10:05 - 10:35	COFFEE BREAK		
10:35 - 11:25	**Boţ**	**Grad**	**G.-Falset**
11:25 - 12:15	**Oliva**	**Ferreira**	**Leuştean**
12:15 - 14:00	LUNCH		
14:00 - 14:50	**Nicolae**	**Bačák**	**S.-Altuzarra**
14:50 - 15:40	**Dinis**	**Arthan**	**Saeidi**
15:40 - 16:10	COFFEE BREAK		
16:10 - 16:35	**Dumitru**	**Firmino**	**Pischke**
16:35 - 17:00	**Neri**	**Wan**	
17:00 - 17:50	**Pinto**	**Sanders**	**Kohlenbach**
17:50 - 18:00			Closing Address